

EXAMPLES OF SURFACES WITH CANONICAL MAP OF MAXIMAL DEGREE

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ABSTRACT. It is shown by A. Beauville that if the canonical map $\varphi_{|K_M|}$ of a complex smooth projective surface M is generically finite, then $\deg(\varphi_{|K_M|}) \leq 36$. The first example of such a surface optimizing the inequality was found recently by the second author, arising from a very special fake projective plane. In this article, we generalize the method above, list and classify all surfaces with optimal canonical degree arising from Galois étale coverings of all fake projective planes.

Let M be a smooth complex projective minimal surface of general type with $p_g(M) \neq 0$. Assume that the canonical map,

$$\varphi = \varphi_{|K_M|} : M \dashrightarrow W = \overline{\varphi(M)} \subseteq \mathbb{P}^{p_g(M)-1},$$

is generically finite onto its image $W := \overline{\varphi(M)}$. We are interested in the *canonical degree* of M , the degree of φ . If φ is not generically finite, then we simply say that M has canonical degree zero.

The following proposition is proved in [B], cf. [Y]. We include the proof here for the completeness.

Proposition 0.1. *Let M be a minimal surface of general type whose canonical map $\varphi = \varphi_{|K_M|}$ is generically finite. Then $\deg \varphi \leq 36$. Moreover, $\deg \varphi = 36$ if and only if M is a ball quotient $\mathbf{B}_{\mathbb{C}}^2/\Sigma$ with $p_g(M) = 3$, $q(M) = 0$, and $|K_M|$ is base point free.*

Proof. Suppose that $\varphi : M \dashrightarrow W = \overline{\varphi(M)} \subseteq \mathbb{P}^{p_g-1}$ is generically finite, where $p_g = p_g(M)$. Let P be the mobile part of $|K_M|$. Let $S \rightarrow M$ be a resolution of P and P_S be the induced base point free linear system defining $S \rightarrow W$. Then

$$\deg \varphi \cdot (p_g - 2) \leq \deg \varphi \cdot \deg W = P_S^2 \leq P^2 \leq K_M^2 \leq 9\chi(\mathcal{O}_M) \leq 9(1 + p_g).$$

The first inequality is the degree bound of a non-degenerate surface in \mathbb{P}^n given in [B], while the fourth inequality is the Bogomolov-Miyaoka-Yau inequality. Hence as $p_g \geq 3$, we have

$$\deg \varphi \leq 9\left(\frac{1 + p_g}{p_g - 2}\right) \leq 36.$$

Key words: surface of general type, canonical degree, fake projective planes.

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Moreover, $\deg \varphi = 36$ only when $p_g(M) = 3$, $q(M) = 0$, and $P_S^2 = P^2 = K_M^2$. This is only possible when $|K_M|$ is base point free. In such case, $K_M^2 = 36 = 9\chi(\mathcal{O}_M)$ and hence M is a ball quotient $\mathbf{B}_{\mathbb{C}}^2/\Sigma$ by the results of Aubin and Yau, cf. [B] or [BHPV], \square

Surfaces with low canonical degrees have been constructed, see [P] or [DG] for more references. The first example of a surface with maximal canonical degree 36 is constructed by [Y] as a suitably chosen $C_2 \times C_2$ -Galois cover of a special fake projective plane X . The fake projective plane X in [Y] has $\text{Aut}(X) = C_7 : C_3$, and by [LY] it satisfies $h^0(X, 2L_X) = 0$ for every ample generator L_X of $\text{NS}(X)$. The choice of the lattice for the ball quotient M is explicitly described in [Y] via the classifying data of [PY] and [CS].

Galois étale covers are the same as unramified normal coverings. The purpose of this paper is to generalize the result in [Y] and list all possible surfaces of maximal canonical degree arising from unramified normal coverings of fake projective planes.

Theorem 0.2. *Let $M \rightarrow X$ be a Galois étale cover of degree four to a fake projective plane X . If $q(M) = 0$, then M has canonical degree 36.*

We observe that there are many degree four Galois étale covers of fake projective planes with maximal canonical degree 36. A degree four Galois étale cover $M \rightarrow X$ of a fake projective plane X is determined by a quotient of $H_1(X, \mathbb{Z})$ of order four. We list all the fake projective planes with a quotient $H_1(X, \mathbb{Z}) \twoheadrightarrow C_2 \times C_2$ or $H_1(X, \mathbb{Z}) \twoheadrightarrow C_4$, to be explained in details in Lemma 1.1 of Section 1. Since each class consists of two surfaces via conjugate complex structures, there are 54 surfaces (up to biholomorphism) where there is a degree four Galois étale cover, listed in the table below.

Corollary 0.3. *The number of lattices associated to surfaces of maximal canonical degree arising from four fold Galois étale covers of the fake projective planes are listed by N_1 in the table below. In particular, there are altogether 835 such lattices arising in this way. This gives rise to 1670 non-biholomorphic smooth surfaces of maximal canonical degree.*

In the table below, only lattices of fake projective planes giving rise to a Galois étale cover of degree four are listed, which is the case if there is a subgroup of index four in the lattice Π corresponding to a given fake projective plane $X = \mathbf{B}_{\mathbb{C}}^2/\Pi$. The listing of the fake projective planes follows the conventions in [PY] and [CS]. The entry N_0 gives the number of subgroups of index four of the lattice Π . The entry N_1 gives the number of normal subgroups $\Sigma = \pi_1(M)$ in Π which satisfies the properties that $\Sigma^{\text{ab}} = H_1(M, \mathbb{Q}) = 0$. Note that by Poincaré duality, this last equality implies $q(M) = 0$.

(k, ℓ, \mathcal{T})	class	X	$\text{Aut}(X)$	$H_1(X, \mathbb{Z})$	N_0	N_1
$(\mathbb{Q}, \mathbb{Q}(\sqrt{-1}), \{5\})$	$(a = 1, p = 5, \emptyset)$	$(a = 1, p = 5, \emptyset, D_3)$	C_3	$C_2 \times C_4 \times C_{31}$	4	3
	$(a = 1, p = 5, \{2\})$	$(a = 1, p = 5, \{2\}, D_3)$	C_3	$C_4 \times C_{31}$	4	1
$(\mathbb{Q}, \mathbb{Q}(\sqrt{-1}), \{2, 5\})$	$(a = 1, p = 5, \{2I\})$	$(a = 1, p = 5, \{2I\})$	$\{1\}$	$C_2 \times C_3 \times C_4^2$	47	19
$(\mathbb{Q}, \mathbb{Q}(\sqrt{-2}), \{3\})$	$(a = 2, p = 3, \emptyset)$	$(a = 2, p = 3, \emptyset, D_3)$	C_3	$C_2^2 \times C_{13}$	4	1
	$(a = 2, p = 3, \{2\})$	$(a = 2, p = 3, \{2\}, D_3)$	C_3	$C_2^2 \times C_{13}$	4	1
$(\mathbb{Q}, \mathbb{Q}(\sqrt{-2}), \{2, 3\})$	$(a = 2, p = 3, \{2I\})$	$(a = 2, p = 3, \{2I\})$	$\{1\}$	$C_2^4 \times C_3$	83	35
$(\mathbb{Q}, \mathbb{Q}(\sqrt{-7}), \{2\})$	$(a = 7, p = 2, \emptyset)$	$(a = 7, p = 2, \emptyset, D_3 2_7)$	$C_7 : C_3$	C_2^4	91	35
		$(a = 7, p = 2, \emptyset, 7_{21})$	$\{1\}$	$C_2^2 \times C_3 \times C_7$	3	1
	$(a = 7, p = 2, \{7\})$	$(a = 7, p = 2, \{7\}, D_3 2_7)$	$C_7 : C_3$	C_2^3	7	7
		$(a = 7, p = 2, \{7\}, D_3 7_7^1)$	C_3	$C_2^2 \times C_7$	2	1
		$(a = 7, p = 2, \{7\}, 7_{21})$	$\{1\}$	$C_2^3 \times C_3$	19	7
$(\mathbb{Q}, \mathbb{Q}(\sqrt{-7}), \{2, 3\})$	$(a = 7, p = 2, \{3\})$	$(a = 7, p = 2, \{3\}, D_3)$	C_3	$C_2 \times C_4 \times C_7$	4	3
		$(a = 7, p = 2, \{3\}, 3_3)$	$\{1\}$	$C_2^2 \times C_3 \times C_4$	19	11
	$(a = 7, p = 2, \{3, 7\})$	$(a = 7, p = 2, \{3, 7\}, D_3)$	C_3	$C_4 \times C_7$	2	1
		$(a = 7, p = 2, \{3, 7\}, 3_3)$	$\{1\}$	$C_2 \times C_3 \times C_4$	7	3
$(\mathbb{Q}, \mathbb{Q}(\sqrt{-7}), \{2, 5\})$	$(a = 7, p = 2, \{5\})$	$(a = 7, p = 2, \{5\})$	$\{1\}$	$C_2^2 \times C_9$	3	1
$(\mathbb{Q}, \mathbb{Q}(\sqrt{-15}), \{2\})$	$(a = 15, p = 2, \emptyset)$	$(a = 15, p = 2, \emptyset, D_3)$	C_3	$C_2^2 \times C_7$	2	1
		$(a = 15, p = 2, \emptyset, 3_3)$	$\{1\}$	$C_2^3 \times C_9$	11	7
		$(a = 15, p = 2, \{3\})$	C_3	$C_2^3 \times C_3$	19	7
	$(a = 15, p = 2, \{5\})$	$(a = 15, p = 2, \{5\}, 3_3)$	$\{1\}$	$C_2^2 \times C_9$	3	1
	$(a = 15, p = 2, \{3, 5\})$	$(a = 15, p = 2, \{3, 5\}, 3_3)$	C_3	$C_2^2 \times C_3$	1	1
$(\mathcal{C}_{18}, \{v_3\})$	$(\mathcal{C}_{18}, p = 3, \emptyset)$	$(\mathcal{C}_{18}, p = 3, \emptyset, d_3 D_3)$	$C_3 \times C_3$	$C_2^2 \times C_{13}$	1	1
$(\mathcal{C}_{20}, \{v_2\})$	$(\mathcal{C}_{20}, \{v_2\}, \emptyset)$	$(\mathcal{C}_{20}, \{v_2\}, \emptyset, D_3 2_7)$	$C_7 : C_3$	C_2^6	651	651
	$(\mathcal{C}_{20}, \{v_2\}, \{3+\})$	$(\mathcal{C}_{20}, \{v_2\}, \{3+\}, D_3)$	C_3	$C_4 \times C_7$	2	1
		$(\mathcal{C}_{20}, \{v_2\}, \{3+\}, \{3+\}_3)$	$\{1\}$	$C_2 \times C_3 \times C_4$	7	3
	$(\mathcal{C}_{20}, \{v_2\}, \{3-\})$	$(\mathcal{C}_{20}, \{v_2\}, \{3-\}, D_3)$	C_3	$C_4 \times C_7$	2	1
		$(\mathcal{C}_{20}, \{v_2\}, \{3-\}, \{3-\}_3)$	$\{1\}$	$C_2 \times C_3 \times C_4$	7	3

TABLE 1

We remark that the third column of the above table contains the list of all fake projective planes on which an unramified normal covering of order 4 exists. The order of the covering is dictated by the possible existence of a surface of maximal canonical degree, i.e., $K_M^2/K_X^2 = 4$.

Our results have implications on the optimal canonical degree for smooth threefolds of general type. We refer the readers to Section 4 for more details.

Corollary 0.4. *There exists many examples of smooth threefolds of general type Y with the degree of canonical map $\Phi_{|K_Y|}$ satisfying $\deg(\Phi_{|K_Y|}) = 72$. In fact, there exists such threefolds with $p_g(Y) = 3g$ and $K_Y^3 = 72(g-1)$ for each $g \geq 2$.*

Note that both Theorem 0.2 and Corollary 0.4 are proved without relying on data from computer implementation or results from [CS]. The need of such data is required only in getting the precise listing of possible surfaces in Corollary 0.3 and Table 1.

Comparing to the result in [Y], we have to deal with several difficulties to classify surfaces of maximal canonical degree as achieved above. In the first place, the surface studied in [Y] has Picard number one, which is a deep result in automorphic forms from [R], [BR], and is used in [Y] to simplify the geometric arguments. For a general normal covering of a fake projective plane of degree four, it is not clear that the Picard number is equal to one. In the second place, the argument of [Y] makes use of the fact that the covering group of the candidate surface over the corresponding fake projective plane is $C_2 \times C_2$. Generator of each of the C_2 factors was used in the argument there. In the general situation studied here, the covering group of the candidate surface over the corresponding fake projective plane may be C_4 or $C_2 \times C_2$. The argument of [Y] was used for part of the argument for the case of $C_2 \times C_2$. An alternative argument is devised for the case of C_4 in this paper.

To find which étale cover works, as a first step we list all normal subgroups of index four in a lattice associated to a fake projective plane. All fake projective planes supporting such a subgroup was listed in the third column of Table 1 above. Now for each of surfaces listed, we exhaust all possible normal subgroups of index 4. The procedure of finding such a surface as well as verification of necessary conditions stated in Theorem 0.2 is given for an explicit fake projective plane in Section 5. The same procedure is carried over for all cases listed in the third column of Table 1. The explicit computation is accomplished by using Magma. The main part of the paper is to show that each such surface is a surface of maximal canonical degree as stated in Theorem 0.2.

Here is the organization of this paper. We first list some preliminary results crucial to our construction. In Section 2, we establish generic finiteness of the canonical map. Then we prove base point freeness of the canonical map for $C_2 \times C_2$ and C_4 normal coverings in sections 3. The construction of M with irregularity $q(M) = 0$ is given in Section 4. Finally we remark on the corresponding problem in dimension 3 in Section 5.

Though this paper, linear equivalence and numerical equivalence of divisors are written respectively as $D_1 \sim D_2$ and $D_1 \equiv D_2$. The cyclic group of order n is denoted by C_n .

1. PRELIMINARY

Let $X = \mathbf{B}_{\mathbb{C}}/\Pi$ be a fake projective plane with $\pi_1(X) = \Pi$. It is known from definition that the first Betti number of X is trivial. According to [PY], there is always a nontrivial torsion element in $H_1(X, \mathbb{Z})$. The torsion group $H_1(X, \mathbb{Z})$ is available from [CS].

Lemma 1.1. *A fake projective plane X has an unramified normal covering of degree four if and only if there is a quotient group of order four of $H_1(X, \mathbb{Z})$.*

Proof. We know that $H_1(X, \mathbb{Z})$ is a direct sum of finite cyclic abelian groups, since the first Betti number of X is trivial. If Q is a quotient group of order four of $H_1(X, \mathbb{Z})$, then there is a homomorphism

$$\rho : \Pi \rightarrow \Pi/[\Pi, \Pi] = H_1(X, \mathbb{Z}) \rightarrow Q.$$

The kernel of ρ gives rise to a normal subgroup Σ of index four in Π , with Q as the deck transformation group of the covering map $M = \mathbf{B}_{\mathbb{C}}^2/\Sigma \rightarrow X = \mathbf{B}_{\mathbb{C}}^2/\Pi$.

On the other hand, if there is a normal subgroup Σ of index four in Π , it leads to a homomorphism $\sigma : \Pi \rightarrow \Pi/\Sigma$. As a group of order four is always Abelian, σ factors through a homomorphism $\Pi/[\Pi, \Pi] \rightarrow \Pi/\Sigma$. We conclude that Π/Σ lives as a quotient group of order four of $\Pi/[\Pi, \Pi] = H_1(X, \mathbb{Z})$. \square

We consider an étale cover $\pi : M \rightarrow X$ corresponds to a subgroup $\pi_1(M) \leq \Pi$ of index four. In particular, the finite group $\mathcal{G} = \Pi/\pi_1(M)$ is either $C_2 \times C_2$ or C_4 . The following lemmas are explained in [Y] and we include them here for the convenience of the reader.

Lemma 1.2. *Let M be a smooth projective surface and assume that there is an unramified cover $\pi : M \rightarrow X$ of degree four to a fake projective plane X . Suppose that $q(M) = 0$, then $p_g(M) = 3$.*

Proof. Since $\pi : M \rightarrow X$ is étale, $\chi(\mathcal{O}_M) = 4\chi(\mathcal{O}_X) = 4$ as $p_g(X) = q(X) = 0$. It follows that $p_g(M) = 3$ if $q(M) = 0$. \square

Suppose now that we construct a surface M as described in the above lemma. We study the canonical map $\varphi = \varphi|_{K_M} : M \dashrightarrow \mathbb{P}^2$. We will assume that $\pi : M \rightarrow X$ is a *Galois cover*, i.e., $\pi_1(M) \leq \Pi$ is normal. Note that then $|K_M|$ is invariant under the Galois group $\mathcal{G} = \text{Gal}(M/X) = \Pi/\pi_1(M)$. The following lemma is crucial.

Lemma 1.3. *Let M be a smooth projective surface and P be a positive dimensional linear system on M . Suppose that $\mathcal{G} \subseteq \text{Aut}(M)$ is a subgroup of the automorphism group of M and P is \mathcal{G} -invariant, i.e., $g^*P \subseteq P$ for any $g \in \mathcal{G}$. Consider the map $\varphi_P : M \dashrightarrow \varphi(M) \subseteq \mathbb{P}^N$. If \mathcal{G} acts linearly locally around base points of P , then there is an induced action of \mathcal{G} on $W = \overline{\varphi(M)}$.*

Proof. Let $\widehat{M} \rightarrow M$ be a composition of finitely many blow-ups of smooth points that resolves the map $\varphi : M \dashrightarrow \mathbb{P}^N$. Let $\widehat{\varphi} : \widehat{M} \rightarrow W = \widehat{\varphi}(\widehat{M}) = \overline{\varphi(M)} \subseteq \mathbb{P}^N$ be the induced morphism. Since \mathcal{G} acts linearly locally around base points of P , there is an induced action of \mathcal{G} on \widehat{M} .

To show that there is an induced action of \mathcal{G} on W , consider $z \in \mathbb{P}^N$ so that $z = \widehat{\varphi}(x)$ for some $x \in \widehat{M}$. We define the action of $\gamma \in \mathcal{G}$ on z by

$$\gamma \cdot z := \widehat{\varphi}(\gamma \cdot x).$$

To show this is well-defined, we assume that there are $x, y \in \widehat{M}$ with $\widehat{\varphi}(x) = \widehat{\varphi}(y)$. If $\widehat{\varphi}(-) = [s_0(-) : s_1(-) : \cdots : s_N(-)]$, then there is a $k \in \mathbb{C}^*$ such that $s_i(x) = ks_i(y)$ for $i = 0, 1, \dots, N$. Since $\gamma^*s_i \in P$ as P is \mathcal{G} -invariant and $P = \langle s_0, s_1, \dots, s_N \rangle$, we can write $\gamma^*s_i = a_{i0}s_0 + a_{i1}s_1 + \cdots + a_{iN}s_N$ for some constants a_{ij} . It is now easy to see that

$$\begin{aligned} \widehat{\varphi}(\gamma \cdot x) &= [s_0(\gamma \cdot x) : s_1(\gamma \cdot x) : \cdots : s_N(\gamma \cdot x)] \\ &= [\gamma^*s_0(x) : \gamma^*s_1(x) : \cdots : \gamma^*s_N(x)] \\ &= [\gamma^*s_0(y) : \gamma^*s_1(y) : \cdots : \gamma^*s_N(y)] \\ &= \widehat{\varphi}(\gamma \cdot y). \end{aligned}$$

□

Let $\pi : M \rightarrow X$ be a Galois étale cover of degree four of a fake projective plane X with Galois group $\mathcal{G} = \text{Gal}(M/X)$. Since M is a ball quotient, the group \mathcal{G} acts locally linearly on any point of M . In particular, Lemma 1.3 applies to M . Recall that $\mathcal{G} = \text{Gal}(M/X)$ is either $C_2 \times C_2$ or C_4 . The following is quoted from [W]. Let $\mathcal{G} < \mathcal{G}^+(M)$ be a cyclic group generated by an element g of finite order n . Here $\mathcal{G}^+(M)$ is the group of all homeomorphisms acting trivially on homology, endowed with the compact-open topology. By Smith theory, each connected component of the fixed point set $M^{\mathcal{G}}$ is a homology \mathbb{P}^k with $k = 0$ or 1 . Since $\chi(M^{\mathcal{G}}) = L(g) = 3$, $M^{\mathcal{G}}$ consists of three isolated points or consists of a single point and a 2-sphere.

Lemma 1.4. *If $\mathcal{G} = C_n$ acts on \mathbb{P}^2 , the fixed point set $\text{Fix}(\mathcal{G})$ consists of either a point and a disjoint \mathbb{P}^1 (type I), or three isolated points (type II). Moreover, if $n = 2$, only fixed point set of type I can occur.*

Proof. The last statement is a theorem in [B, pp. 378, Theorem 3.1]: If $\mathcal{G} = C_p$ with p prime is acting on \mathbb{P}^n , then the number of components of $\text{Fix}(\mathcal{G})$ is at most p . □

Also recall the following “negativity lemma.”

Lemma 1.5. *Let M be a smooth projective surface. Suppose that $\varphi : M \rightarrow M'$ is a generically finite morphism. If $F \neq 0$ is an effective divisor on M and $F^2 \geq 0$, then $\varphi(F)$ is positive dimensional.*

Proof. Replace by the Stein factorization, we can assume that φ is birational. If $\varphi(F)$ is zero dimensional, then by Hodge index theorem $F^2 < 0$ unless $F = 0$. □

2. GENERIC FINITENESS

The following proposition generalizes Proposition 1 in [Y], where the author makes the assumption that $\rho(M) = 1$.

Proposition 2.1. *Let X be a fake projective plane. Suppose that there is a Galois étale cover $\pi : M \rightarrow X$ of degree four and $q(M) = 0$, then the canonical map $\varphi : M \dashrightarrow \mathbb{P}^2$ is generically finite.*

Proof. From Lemma 1.2, we know that $p_g(M) = 3$ and hence the canonical map maps to \mathbb{P}^2 . Decompose $|K_M| = P + F$, where P is the mobile part and F is the fixed divisor. By construction, we have $\varphi = \varphi_{|K_M|} = \varphi_P : M \dashrightarrow \mathbb{P}^2$. We will abuse the notation: P will be the mobile linear system or a general member in it.

Assume that $\overline{\varphi(M)} = C \subseteq \mathbb{P}^2$ is a curve. We will derive a contradiction.

First of all, we claim that P is not base point free, or equivalently $P^2 \neq 0$. Assume now $P^2 = 0$. We consider $\mathcal{G} = \text{Gal}(M/X)$. Since $g^*K_M = K_M$ for any $g \in \mathcal{G}$, we have that $g^*F = F$ for each $g \in \mathcal{G}$. Indeed, g^*P is a mobile sub-linear system of $|K_M|$ and hence $g^*F \geq F$ as Weil divisors. Hence as π is Galois, $F = \pi^*F_X$ for an effective divisor F_X on X . Moreover, if $\text{NS}(X) = \langle L_X \rangle$ for an ample divisor L_X , then $K_X \equiv 3L_X$, $F_X \equiv lL_X$ for some $0 \leq l \leq 3$, and $P \equiv \pi^*(3-l)L_X$. Now, $P^2 = 0$ implies that $l = 3$ and hence $P \equiv 0$. This is a contradiction as a non-zero effective

divisor cannot be numerically trivial. We remark here that even though $q(X) = 0$ for a fake projective plan X , X may still have non-trivial torsion line bundles.

Since $\varphi : M \dashrightarrow C \subseteq \mathbb{P}^2$ is not a morphism, we take a composition of finitely many smooth blow-ups $\rho : S \rightarrow M$ to resolve P and let $\psi : S \rightarrow C \subseteq \mathbb{P}^2$ be the induced morphism. We have the following diagram after taking the Stein factorization of $\psi : S \rightarrow C$:

$$\begin{array}{ccc} S & \xrightarrow{\beta} & \tilde{C} \\ \rho \downarrow & \searrow \psi & \downarrow \alpha \\ M & \dashrightarrow_{\varphi} & C \subseteq \mathbb{P}^2 \end{array}$$

If $\rho^*P = P_S + F_S$, where $P_S = \psi^*|\mathcal{O}_C(1)|$ is base point free, $F_S \geq 0$ is the fixed divisor, and $\psi = \psi_{P_S}$, then F_S is a non-trivial effective ρ -exceptional divisor with $\beta(F_S) = \tilde{C}$. In particular, $\tilde{C} \cong \mathbb{P}^1$ as all the components of F_S are rational. Since $\alpha : \tilde{C} \rightarrow C$ is defined by $\alpha^*|\mathcal{O}_C(1)| \subseteq |\mathcal{O}_{\mathbb{P}^1}(d)|$ for some $d \geq 1$ and hence an element in P_S is given by β^*H for some $H \in |\mathcal{O}_{\mathbb{P}^1}(d)|$, we have $P_S \supseteq \beta^*|\mathcal{O}_{\mathbb{P}^1}(d)|$. In particular, we get

$$P_S = \psi^*|\mathcal{O}_C(1)| = \beta^*\alpha^*|\mathcal{O}_C(1)| = \beta^*|\mathcal{O}_{\mathbb{P}^1}(d)|.$$

As $\dim P_S = p_g(M) = 3$, we get $d = 2$ and $C \subseteq \mathbb{P}^2$ being irreducible and non-degenerate is a smooth quadratic curve in \mathbb{P}^2 .

Let $S_{c'}$ be a general fiber of $S \rightarrow \tilde{C}$ and $D = \rho_*(S_{c'}) \equiv P/2$ be the corresponding divisor. Note that $d = 2$ and hence D is a prime divisor. Recall that $\pi : M \rightarrow X$ is Galois, $K_M = \pi^*K_X \equiv \pi^*(3L_X)$ and $P \equiv \pi^*(lL_X)$ for some $1 \leq l \leq 3$ as $P^2 \neq 0$, where $\text{NS}(X) = \langle L_X \rangle$ and $L_X^2 = 1$. It follows from the genus formula,

$$(K_M + D) \cdot D = 2g_a(D) - 2 \in 2\mathbb{Z},$$

that $l = 2$ is the only possibility. Hence $P \equiv \pi^*(2L_X)^1$ and $F = \pi^*F_X \equiv \pi^*L_X$.

By Lemma 1.3, \mathcal{G} acts on $C \cong \mathbb{P}^1$ holomorphically. From the Lefschetz fixed point formula, \mathcal{G} has two fixed points on C . Let $G_1 \sim G_2$ be two distinct \mathcal{G} -fixed divisors in P corresponding to the two fixed points on C . In particular, G_i 's are \mathcal{G} -invariant and hence descend to effective divisors $G_i^X \equiv 2L_X$ on X . Consider the rational map $X \dashrightarrow \mathbb{P}^1$ defined by the pencil $V = \langle \lambda G_1^X + \mu G_2^X \rangle$. This rational map cannot be a morphism since $\rho(X) = 1$. Let $\sigma : Y \rightarrow X$ be a composition of smooth blow-ups that resolves this rational map. We can assume that $Y \rightarrow \mathbb{P}^1$ is relative minimal. Note that a general element $G_X \in V$ is connected: Consider the exact sequence

$$\mathbb{C} \cong H^0(X, \mathcal{O}_X) \rightarrow H^0(G_X, \mathcal{O}_{G_X}) \rightarrow H^1(X, \mathcal{O}_X(-G_X)),$$

¹If $h^0(X, 2L_X) = 0$ for any ample generator L_X on X , then we arrive the required contradiction. This is exactly the argument in [Y], where the vanishing holds for X a very special fake projective plane as discussed in the introduction.

where the last term $H^1(X, \mathcal{O}_X(-G_X)) \cong H^1(X, \mathcal{O}_X(K_X + G_X))^\vee = 0$ by the Kodaira vanishing theorem as $G_X \equiv 2L_X$ is ample. Hence the morphism $Y \rightarrow \mathbb{P}^1$ has connected fibers. In particular, a general fiber Y_b is smooth, where $Y_b = \sigma_*^{-1}(G_X)$ is also the proper transform of G_X . Hence G_X is irreducible with $Y_b = (G_X)^\nu$ the normalization and

$$6 = \frac{1}{2}(K_X + G_X).G_X + 1 = p_a(G_X) = g(Y_b) + h^0(\delta),$$

where $\delta = \sigma_*\mathcal{O}_{Y_b}/\mathcal{O}_{G_X}$ is a torsion sheaf supported on $\text{Sing}(G_X)$.

Write $K_Y = \sigma^*K_X + E$ for some effective σ -exceptional divisor E , then

$$2g(Y_b) - 2 = K_Y.Y_b = K_X.G_X + E.Y_b \geq 6 + 1,$$

where $E.Y_b \geq 1$ is by construction. Hence $5 \leq g(Y_b) \leq 6$. We claim that neither cases can happen.

If $g(Y_b) = 6$, then $G_X \cong Y_b$ is smooth and $G_M = \pi^{-1}(G_X)$ is either a disjoint union of smooth curves or is an irreducible smooth curve. On the other hand, $\pi^*G_X \in P$ and a member of P is connected by the same argument as before. But by construction, a general member of P must be reducible with two components coming from two fibers of $S \rightarrow \tilde{C}$. It follows that G_M must be singular. This is a contradiction. In the next step, we rule out $g(Y_b) = 5$ case.

Write $\sigma^*G_X = Y_b + \sum_{i \in I} E_i$, where $E_i \cong \mathbb{P}^1$ is σ -exceptional and here we allow $E_i = E_j$ for $i \neq j$ in $\sum_{i \in I} E_i$. From $\sigma_*(Y_b) = G_X \equiv 2L_X$, we have

$$(\star) \quad \left(\sum_{i \in I} E_i\right)^2 = (\sigma^*G_X - Y_b)^2 = (2L_X)^2 - 2(2L_X).(2L_X) = -4,$$

Since $E_i^2 \leq -1$, we have $K_Y.E_i \geq -1$ and

$$6 = K_X.G_X = K_Y.\sigma^*G_X = K_Y.Y_b + \sum_{i \in I} (K_Y.E_i) \geq 2g(Y_b) - 2 - |I|.$$

On the other hand, a single blow up drops the self intersection number of the proper transform of G_X by at least one. As $G_X^2 = 4$ and $Y_b^2 = 0$, combined with the above inequality we get $2g(Y_b) - 8 \leq |I| \leq 4$.

Suppose now $g(Y_b) = 5$. Then $2 \leq |I| \leq 4$, $(\sum_{i \in I} E_i)^2 = -4$, and $\sum_{i \in I} (K_Y.E_i) = -2$. Since $(K_Y + E_i).E_i = -2$ for all i , we obtain

$$-6 = \left(\sum_{i \in I} E_i\right)^2 + \sum_{i \in I} (K_Y.E_i) = -2|I| + 2 \sum_{i < j} E_i.E_j,$$

and

$$0 \leq \sum_{i < j} E_i.E_j = |I| - 3.$$

Hence $|I| \geq 3$, where $|I| = 3$ only if $E_i.E_j = 0$ for any $i \neq j$. In this case, there are three disjoint E_i 's: two (-1) -curves E_1 and E_2 , and one (-2) -curve E_3 , as $E_i^2 \leq -1$ and $(\sum_i E_i)^2 = \sum_i E_i^2 = -4$. Contracting E_3 gives a nodal point on X and this is absurd. Hence $|I| = 4$.

For $|I| = 4$, there can only be two (-1) -curves and two (-2) -curves in $\sum_i E_i$. Let E_1 and E_2 be (-1) -curves, and E_3 and E_4 be (-2) -curves. By construction, each (-2) -curve must intersect at least one (-1) -curve and hence

$$1 = |I| - 3 = \sum_{i < j} E_i \cdot E_j \geq E_3 \cdot (E_1 + E_2) + E_4 \cdot (E_1 + E_2) \geq 2.$$

This is again absurd.

Hence we conclude that $\dim \overline{\varphi(M)} \neq 1$. Since $\varphi(M) \subseteq \mathbb{P}^2$ has to be positive dimensional, we conclude that $\varphi : M \dashrightarrow \mathbb{P}^2$ must be dominant and hence generically finite. \square

3. BASE POINT FREENESS

Let $M \rightarrow X$ be a Galois étale cover of degree four and $\mathcal{G} = \text{Gal}(M/X)$ be the corresponding Galois group. Suppose that $q(M) = 0$ and hence $p_g(M) = 3$ by Lemma 1.2. We consider two different cases: $\mathcal{G} = C_2 \times C_2$ and $\mathcal{G} = C_4$.

First recall the following fact that we have used in the proof of Proposition 2.1: Decompose $|K_M| = P + F$ into mobile part P and the fixed part F . Then $F = \pi^* F_X$ for an effective divisor F_X on X . Moreover, if $\text{NS}(X) = \langle L_X \rangle$ for an ample divisor L_X , then $K_X \equiv 3L_X$, $F_X \equiv lL_X$ for some $0 \leq l \leq 3$, and $P \equiv \pi^*(3 - l)L_X$.

Proposition 3.1. *Let $M \rightarrow X$ be a Galois $C_2 \times C_2$ -cover over a fake projective plane X . If $q(M) = 0$, then the canonical map $\varphi_{|K_M|} : M \dashrightarrow \mathbb{P}^2$ is a generically finite morphism of degree 36.*

Proof. From Proposition 2.1, φ is generically finite. Recall that from Lemma 1.3, φ is a \mathcal{G} -map and $F = \pi_X^* F_X$ is \mathcal{G} -invariant, where $F_X \equiv lL_X$ with $l \geq 0$. If $F \neq 0$, i.e., $l > 0$, then the \mathcal{G} -invariant set $\varphi(F)$ must be positive dimensional by Lemma 1.5. On the other hand, consider the action of two C_2 factors of \mathcal{G} . By [HL], the \mathcal{G} -invariant set on \mathbb{P}^2 is isolated and this is a contradiction. We conclude that $l = 0$ and $F = 0$. The same argument as in [Y] then shows that $|K_M| = P$ has no isolated base points and hence is base point free. It is now easy to see that $\deg(\varphi_{|K_M|}) = K_M^2 = 36$. \square

Proposition 3.2. *Let $M \rightarrow X$ be a Galois étale C_4 -cover. Then the canonical map $\varphi_{|K_M|} : M \dashrightarrow \mathbb{P}^2$ is a generic finite surjective morphism of degree 36.*

Proof. Decompose $|K_M| = P + F$ as the mobile part P and the fixed part F . We have shown that the map $\varphi_{|K_M|} = \varphi_P$ is generically finite in Proposition 2.1. Let $\mathcal{G} = \text{Gal}(M/X) = C_4 = \langle a \rangle$. By Lemma 1.4, we consider two possibilities of $\text{Fix}(a)$: type I or type II.

Suppose that $\text{Fix}(a)$ is of type I, i.e., it consists of a fixed line and an isolated fixed point. Let l_a be the unique fixed line of \mathcal{G} on \mathbb{P}^2 . The hyperplane l_a corresponds to a section $s_M \in P$ which is \mathcal{G} -invariant. In particular, s_M descends to a section s_X on X . Hence as $K_M \sim [s_M] + F$, we have $\pi^* K_X \sim \pi^*([s_X] + F_X)$. But a nonzero section in $H^0(M, \pi^*(K_X - ([s_X] + F_X)))$ is just a nonzero constant, which descends to X . Hence $K_X \sim [s_X] + F_X \geq 0$ and this contradicts to $p_g(X) = 0$.

Suppose that $\text{Fix}(a)$ is of type II, i.e., it consists of three isolated fixed points. Let $\rho : S \rightarrow M$ be a composition of blow-ups that resolves φ_P . Then the set of ρ -exceptional divisor is \mathcal{G} -invariant and must be contracted into the zero dimensional set $\text{Fix}(a)$. This means that we can take $S = M$ and hence P is base point free. Now consider the fixed divisor F . As F is \mathcal{G} -invariant and φ is a \mathcal{G} -morphism, $\varphi(F) \subseteq \text{Fix}(a)$ is zero dimensional. By Lemma 1.5, $F^2 < 0$ if $F \neq 0$. On the other hand, as $F \equiv \pi^*F_X$ with $F_X \equiv lL_X$ for some $l \geq 0$, we have $F^2 \geq 0$. Hence it is only possible that $F = 0$ and $|K_M| = P$ is base point free. Clearly, $\deg(\varphi|_{K_M}) = K_M^2 = 36$ as in Proposition 3.1. \square

4. CONSTRUCTION OF M

The arguments of the previous section work for all unramified normal coverings of fake projective planes unconditionally, once the condition that the first Betti number of the covering is trivial is verified. This latter fact is known if the lattice involved is a congruence subgroup of lattice associated to the fake projective plane, following the work of Rogawski [R] and the results of [PY] on the defining division algebras, see also [CS]. In general, we can use the presentation of the lattice associated to fake projective planes from [CS] to achieve this end. The purpose of this section is to give some details about how the first Betti number can be checked in the general case.

First of all, the deck transformation group Π/Σ is either $C_2 \times C_2$ or C_4 . An example of $\Pi/\Sigma = C_2 \times C_2$ is already given in [Y]. Hence we would present an example with $\Pi/\Sigma = C_4$ in detail first.

Consider a fake projective plane with Π denoted by $(C_{20}, \{v_2\}, 3+, D_3)$ in the third column of Table 1. A presentation in terms of generators and relations is given in [CS] by

$$\begin{aligned} \bar{\Gamma} := \langle a, b, c \mid & b^3, a^{-2}bc^2bc^2ac^{-1}b, caba^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}c^{-1}b^{-1}a^{-1}, \\ & a^{-1}cac^2ac^{-1}abca^{-1}c^{-2}b^{-1}, c^{-1}abca^{-1}c^{-1}aca^{-1}c^{-2}b^{-1}a^2, \\ & ab^{-1}c^2a^{-1}c^{-2}b^{-1}a^3bab, bc^2ac^{-3}b^{-1}a^2c^2ac^{-1}, (bc^3a^{-1})^3, \\ & c^{-1}ac^{-1}ba^{-2}ca^{-1}c^{-2}a^{-1}c^{-3}b^{-1}a, \\ & a^{-1}ca^{-1}c^{-2}ba^{-1}ca^{-1}c^{-3}b^{-1}a^{-1}ca^{-1}, \\ & a^{-1}c^{-1}abc^{-1}b^{-1}a^{-1}ca^{-1}bcba^{-1}bca^{-1}c^{-1}, \\ & bc^{-1}b^{-1}a^{-1}ca^{-1}bca^{-1}c^{-1}b^{-1}a^{-1}c^{-1}b^{-1}a^{-1}ca^{-1}bc^2, \\ & a^{-1}ca^{-2}c^{-2}b^{-1}a^2caba^{-2}ca^{-1}c^{-2}a^{-2}, (ba^{-1}b^{-1}a^{-2}bc)^3, \\ & c^{-1}b^{-1}a^{-1}c^{-2}b^{-1}a^2cac^{-1}b^{-1}ab^{-1}a^{-1}c^{-1}b^{-1}a^{-1}b^{-1}a^{-1}ca, \\ & ba^{-1}c^{-1}ac^{-1}ba^{-1}babca^{-1}c^{-1}abc^{-1}b^{-1}a^{-1}b^{-1}a^{-1}cac \rangle \end{aligned}$$

The lattice associated to the fake projective plane is denoted by Π and is generated by the subgroup of index three in $\bar{\Gamma}$ with generators given by

$$\Pi = \langle \bar{\Gamma} \mid a, c, bab^{-1}, bcb^{-1} \rangle.$$

Denote by g_1, \dots, g_4 the elements listed above. With the help of Magma command `LowIndexSubgroups`, we find that there are altogether two subgroups of index 4 in

Π , of which only one is a normal subgroup. The unique normal subgroup Σ of index 4 in Π has generators given by

$$g_3g_1^{-1}, g_4g_2^{-1}, g_1^{-1}g_2^{-1}, g_2^{-1}g_1^{-1}, g_3^{-1}g_2^{-1}, g_4^{-1}g_1^{-1}.$$

The corresponding ball quotient is denoted by $M = \mathbf{B}_{\mathbb{C}}^2/\Sigma$. In this case, $N_1 = 1$. Since $H_1(X, \mathbb{Z}) = C_4 \times C_7$, it is clear from Lemma 1.1 that the covering group should be $\Pi/\Sigma \cong C_4$.

From Magma command `AbelianQuotient`, we check that $H_1(M, \mathbb{Z}) = C_4 \times C_7$. Hence we see that the first Betti number of M is trivial.

Proposition 0.1 implies that $\mathbf{B}_{\mathbb{C}}^2/\Sigma$ is a surface with maximal canonical degree. From the above construction, the number of surface of maximal canonical degree is 1.

Proof of Corollary 0.3 We simply apply the above procedure of construction in the last few paragraphs to each of the fake projective plane listed in column 3 of Table 1. From Proposition 0.1, we first need to enumerate all possible surfaces of maximal canonical degree associated to fake projective planes as listed. It turns out that the number of index four subgroups of the lattice Π to a fake projective plane in the table is recorded in the column N_1 in Table 1. This could be seen by considering subgroups of order 4 in $H_1(X, \mathbb{Z})$ as in Lemma 1.1, or by listing index four subgroups of Π from Magma.

Now we claim that all the different sub-lattices of index 4 in Π in Table 1 gives rise to non-isometric complex hyperbolic forms in terms of the Killing metrics on the locally symmetric spaces. For this purpose, we assume that Λ_1 and Λ_2 are two groups obtained from the above procedure and $B_{\mathbb{C}}^2/\Lambda_1$ is isometric to $B_{\mathbb{C}}^2/\Lambda_2$. From construction, Λ_1 and Λ_2 are normal subgroups of index 4 in two lattices Π_1 and Π_2 corresponding to the fundamental groups of fake projective planes. Let $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ be the corresponding maximal arithmetic groups in the respective classes. As $B_{\mathbb{C}}^2/\Lambda_1$ and $B_{\mathbb{C}}^2/\Lambda_2$ are isometric, Λ_1 is conjugate to Λ_2 as discrete subgroups of the same algebraic group G with $G \otimes \mathbb{R} \cong PU(2, 1)$. Hence the two corresponding maximal lattices $\bar{\Gamma}_1 \cong \bar{\Gamma}_2$, similarly $\Pi_1 \cong \Pi_2$. It follows that they have to come from the same row in the Table 1 and hence correspond to the same subgroup of index 4 in the same lattice associated to some fake projective plane. Hence there are altogether 835 non-isometric complex two ball quotients obtained in this way, by summing over the column of N_1 in Table 1.

Now for each locally symmetric space $M = B_{\mathbb{C}}^2/\Lambda$ obtained as above, it gives rise to a pair of complex structures J_1, J_2 which are conjugate to each other. These two complex structures give rise to two non-biholomorphic complex surfaces $N_1 = (M, J_1)$ and $N_2 = (M, J_2)$. In fact, if they are biholomorphic, the corresponding four-fold quotient $N_1/[\Pi, \Lambda]$ and $N_2/[\Pi, \Lambda]$ are biholomorphic and are fake projective space. This contradicts the results in [KK], see also the Addendum of [PY], that conjugate complex structures on a fake projective space give rise to two different complex structures.

In general, let (M_1, J_1) and (M_2, J_2) be two complex ball quotients obtained from unramified four fold covering of some possibly different fake projective planes. If

(M_1, J_1) and (M_2, J_2) are biholomorphic, they are isometric with respect to the corresponding Bergman (Killing) metrics. Hence from the earlier argument, M_1 is isometry to M_2 and we may regard $M_1 = M_2$. Now the argument of the last paragraph implies that $J_1 = J_2$. In conclusion, we conclude that the 1670 complex surfaces obtained from the pair of conjugate complex structures on the 835 underlying locally symmetric structures give rise to distinct complex surfaces. This concludes the proof of Corollary 0.3. \square

5. REMARK ON MAXIMAL CANONICAL DEGREE OF THREEFOLDS

Theorem 1 has some implications for the optimal degree of the canonical map of threefolds as well. The purpose of this section is to explain literatures in this direction and relations to Theorem 1.

From this point on, consider Y a Gorenstein minimal complex projective threefold of general type with locally factorial terminal singularities. Suppose that the linear system $|K_Y|$ defines a generically finite map $\Phi = \Phi_{|K_Y|} : Y \dashrightarrow \mathbb{P}^{p_g(Y)-1}$. According to [Hac], M. Chen asked if there is an upper bound of the degree of Φ ? A positive answer is provided by Hacon in [Hac] that $\deg(\Phi) \leq 576$. In [C], Cai proved that if one further assumes geometric genus $p_g(Y) > 10541$, then actually $\deg(\Phi) \leq 72$.

The question now is whether the optimal degree can be achieved. The following is a corollary of Theorem 1 and the above discussion.

Proof of Corollary 0.4. Equipped with Theorem 1, the corollary follows essentially from an observation of [C], Section 3.

Take C a smooth hyperelliptic curve of genus $g \geq 2$, then the canonical map $\varphi_{|K_C|} : C \rightarrow \mathbb{P}^{g-1}$ is the composition of the double cover $C \rightarrow \mathbb{P}^1$ with the $(g-1)$ -Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}$. In particular, $\deg(\varphi_{|K_C|}) = 2$, cf. [Har]. Take M a surface satisfying the optimal degree bound $\deg(\varphi_{|K_M|}) = 36$ as in Theorem 1, then $\varphi = \varphi_{|K_M|} : M \rightarrow \mathbb{P}^2$ is a generically finite morphism of $\deg(\varphi) = K_M^2 = 36$.

Now take $Y = X \times C$, then Y is a smooth projective threefold of general type with $p_g(Y) = 3g$ and $\Phi = \Phi_{|K_Y|} : Y \rightarrow \mathbb{P}^{3g-1}$ a morphism. From our construction, it follows that Φ is generically finite and

$$K_Y^3 = 3K_X^2 \cdot K_C = 3 \cdot 36 \cdot (2g-2) = \deg \Phi \cdot \deg W,$$

where $W = \Phi(Y)$ is the Veronese embedding $\mathbb{P}^2 \times \mathbb{P}^{g-1} \hookrightarrow \mathbb{P}^{3g-1}$ defined by $\mathcal{O}(1, 1)$ and $\deg W = 3(g-1)$. Hence $\deg(\Phi) = 72$. \square

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